LETTER TO THE EDITOR

Phase transition in a generalized self-organized depinning model

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Abstract. We investigate numerically a generalized version of a model recently proposed by several authors for describing the critical behaviour of a driven interface in a random medium (the self-organized depinning model). The generalized version allows growth events with simultaneous movements of groups of connected cells of arbitrary size, and includes an external driving force. In $1 + 1$ dimensions, the model exhibits phase transition from a phase with directed percolation exponents to one with the usual exponents ($\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$). But at the transition point, exponents are those of Parisi models.

In recent years the phenomena related to the dynamics of surface growth attracted considerable attention [1, 2]. Many investigations have centred on the dynamic scaling properties of the RMS interface width

$$w(L, t) = [\langle (h'(L, t) - h'(0, 0))^2 \rangle]^{1/2} \sim L^\alpha \cdot \Phi(t/L^{\alpha/\beta}) \tag{1}$$

where $h'(x, t) = h(x, t) - \langle h \rangle$. In general, two kinds of growth process can be distinguished: problems in which the noise term randomly depends on time, and problems in which the noise term is quenched in space. The latter case, which is important in particular for describing the depinning transition of surfaces in random media, seems to be much less understood to date. Parisi [3] has investigated numerically several models which are close to direct enumeration of the continuous equation for the interface position $h(x, t)$:

$$\frac{\partial h}{\partial t} = \Delta h + f(h(x), x) + F \tag{2}$$

where $f(h, x)$ is a quenched random force, and F is an external driving force. The Parisi result for the temporal exponent β at the depinning transition in $1 + 1$ dimensions: $\beta \simeq 0.75$ is consistent with the renormalization group prediction of Nattermann et al [4]: $\beta = 2 - 2\epsilon/9$, with $\epsilon = 4 - d$. On the other hand, Buldyrev et al [5], Tang and Leschhorn [6], Sneppen [7], Zaitsev [8] and Havlin et al [9] have proposed another class of model, relating the critical properties of the depinning transition to the directed percolation problem. Consequently, the exponents α and β at the transition point in these models are mapped to known exponents of the directed percolation problem, and are different from values obtained by Parisi. It was shown recently that in continuous form the models [5–9] can be described by equation (2), but with the nonlinear term $\lambda(\nabla h)^2$, where λ diverges at the depinning transition [15]. Additional interest [10–12] in the model proposed in [7–9] comes from the fact that it provides an interesting example of self-organized critical phenomena, when a system achieves a state with non-trivial power-law correlations without the tuning of any

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external parameters. This is the reason why the model in [7–9] is often referred to as the self-organized depinning (SOD) model.

In this paper we investigate numerically a generalized version of the SOD model in $1 + 1$ dimensions. One purpose of this investigation is to understand how robust the exponents are under natural changes of model parameters, i.e. what is the phase diagram of the generalized model. The more important purpose is to try to understand the fundamental reason behind the difference between the exponents of the Parisi models [3] and the 'directed percolation' models [5–9].

The definition of the SOD model [7–9] is quite simple. To each cell (i, h) on a square lattice, a random pinning force $f(i, h)$ uniformly distributed in the interval $(0, 1)$ is assigned. The interface is specified by a set of integer column heights $h(i)$ $(i = 1, \ldots, L)$. The growth $h(i) \rightarrow h(i) + 1$ proceeds at the site where the pinning force is minimum among all the interface sites. Then adjustments of the neighbouring sites are made until the Kim–Kosterlitz condition [13] $|h(i) - h(i+1)| \leqslant 1$ (for all i) is recovered. In spite of the seeming simplicity, the SOD model leads to a rich set of non-trivial exponents [10], all of which can be related to the exponents of the directed percolation [11, 12].

We note that the 'minimum force' growth rule just summarized seems to be very natural because it reflects the tendency of the system to make the most profitable motion (from the energy point of view). But what should happen if the most profitable motion corresponds in fact not to movement of a single cell, but to some movement of a group of connected cells? More precisely, one can assign to any movement of given group cells $i, i + 1, \ldots, j$ the total energy gain:

$$E(i, j) = f(i, h(i)) + f(i + 1, h(i + 1)) + \cdots + f(j, h(j)) \tag{3}$$

(the energy here is of the same dimension as a force, because all movements are of unit length: $h(i) \rightarrow h(i) \rightarrow h(i) + 1$). It would be reasonable to move the whole group of cells which minimize sum (3) in one time step. Of course, if all the forces are positive as in the case of the original SOD model, the minimum will be always given by movements of a single cell. However, there is, in fact, no reason apart from simplicity to consider all forces to be in the interval $(0, 1)$.

Now we are ready to formulate the generalized version of the SOD model which is the focus of our investigation in this paper. Each cell is assigned by a random force $f(i, h)$ uniformly distributed in the interval $(-\frac{1}{2}, +\frac{1}{2})$, plus a uniform driving force F. At each time step we choose, among all possible groups of connected cells $i, i + 1, \ldots, j$, the group which provides the minimum of the energy E (3) and then move this group. (We note that cells with negative force $f(i, h)$ have in this scheme greater chance to move than those with positive $f(i, h)$. This means one should consider the positive direction of the force to be opposite to the direction of interface propagation.) As in the original SOD model discussed above, each movement of a group is followed by necessary adjustments of neighbouring cells until the Kim–Kosterlitz condition is recovered. As usual, periodic boundary conditions are assumed.

Of course, our modified version of the model is much more challenging for numerical simulations than the original one, because formally we should search for a minimum not among L forces as in the original SOD model, but among L^2 energies (3). In order to make the computations more efficient, we have developed a non-trivial algorithm for searching a minimum of sums of type (3), which needs only an amount of computer time linear in L.

The results from our numerical calculations for the modified SOD model are as follows. The model undergoes a phase transition with variation in the driving force F. For zero or small negative F $(F > F_0 \simeq -0.12)$, the exponents are still the same as for the original
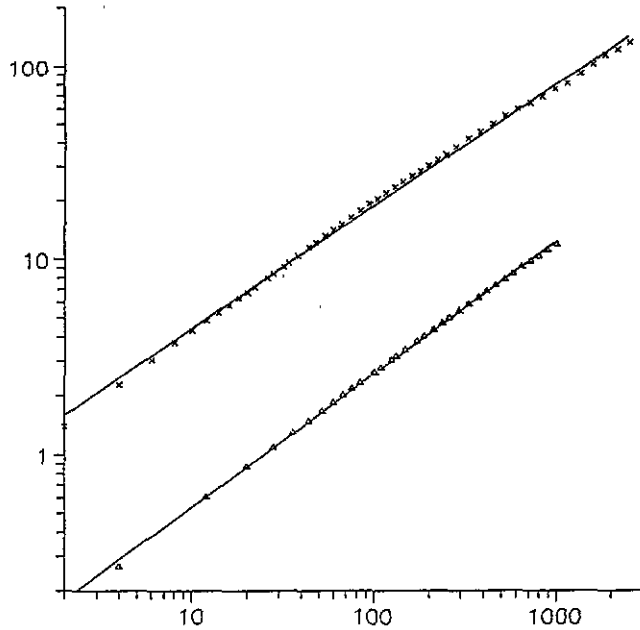
Figure 1. The dependences of the interface width on space (crosses) and time (triangles) for $F > F_0$; which gives exponents $\alpha = 0.63 \pm 0.01$, $\beta = 0.68 \pm 0.01$.
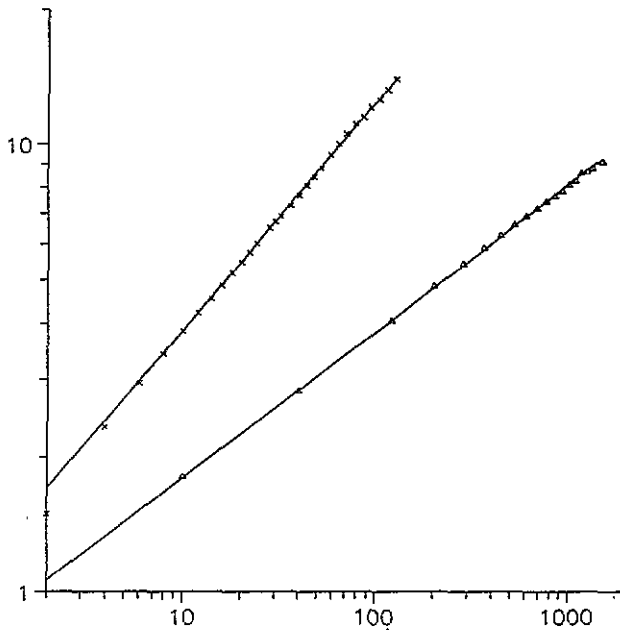

Figure 2. The dependences of the interface width on space (crosses) and time (triangles) for $F < F_0$; with exponents $\alpha = 0.52 \pm 0.02$, $\beta = 0.32 \pm 0.02$.

SOD model. Indeed, the corresponding scaling dependences are shown in figure 1. To be sure that we are in the saturation regime, the first 5×10^6 time steps were omitted from

the measurements. The obtained values of $\alpha = 0.63 \times \pm 0.01$ and $\beta = 0.68 \pm 0.01$ are in good agreement with the known values $\alpha \simeq 0.63$ and $\beta \simeq 0.69$ [5–9]. As F is approaching F_0 from above, the average size of moving group of cells grows to infinity (becomes of the order of the system size L). Simultaneously the average force f_{av}, measured on the surface (which is different from the average force in the bulk), approaches zero. Results for $F < F_0$ (but $F > -0.5$) are shown in figure 2. Here $\alpha = 0.52 \pm 0.02$ and $\beta = 0.32 \pm 0.02$, which is very close to the values $\frac{1}{2}$ and $\frac{1}{3}$ [14]. In this phase the average size of a moving group is only slightly smaller than the system size L, and the difference between them (the size of the non-moving part of the surface) diverges while F approaches F_0 from below. Exponents at the transition point $F = F_0$ (figure 3) $\alpha = 0.98 \pm 0.02$ and $\beta = 0.75 \pm 0.02$ differ from the exponents in both phases above and below F_0. $\beta = 0.75$ is just the Parisi value [3] and α is actually very close to unity [4]. We have not detected any finite interval of F in which these exponents still hold. At $F = -0.5$ the model undergoes another phase transition to the flat phase ($\alpha = 0$), but we are not interested in this particular transition in the present study.
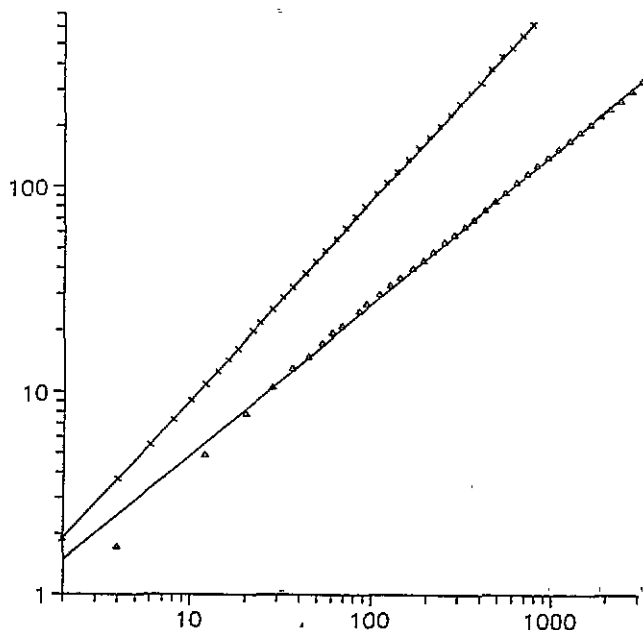

Figure 3. The dependences of the interface width on space (crosses) and time (triangles) for $F = F_0$; with exponents $\alpha = 0.98 \pm 0.02$, $\beta = 0.75 \pm 0.02$.

The fact that the average force f_{av}, measured on the surface, equals zero at the transition point $F = F_0$, seems to be rather natural. Indeed, if we imagine an elastic surface, driven by external force in a random medium, and just write down the Newton equation for the average displacement of the surface, then the total sum of all forces acting on the surface should enter this equation. This total sum is proportional to f_{av}. Therefore one can conclude that the depinning transition of the surface actually corresponds to the condition $f_{av} = 0$. Remarkably, the same results as those presented in figure 3 can also be obtained without tuning F to F_0, but using the self-tuning version of the model, which imitates the condition $f_{av} = 0$. In this version, we calculate f_{av} at each time step, and use $f(i, h) - f_{av}$ instead of $f(i, h)$ for calculating the energies $E(i, j)$ (3).

By definition in our model the surface does not stop motion for any value of the driving force F. The artificial motion in the limit $t \to \infty$ below the depinning transition should be eliminated allowing only profitable movements, decreasing the energy of the system. We accepted the condition that the minimum of the energies (3) counted with the energy of correspondent adjustments should be negative. The calculations show that with such an additional condition the surface indeed finally stops for $F > F_0$. The space exponent α of the pinned surface is found to be the same as without the additional constraint, and the time exponent β stays as a transient one.

In conclusion, we would like to stress the following. The models in Buldyrev *et al* [5] and Tang and Leschhorn [6] describe the depinned phase ($\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$) and the pinned phase with directed percolation exponents. Although some effective α slightly above the transition can seem to be higher than the directed percolation value $\alpha \simeq 0.63$ [5, 6], precisely at the transition $\alpha \simeq 0.63$ as in the pinned phase. The SOD model [7–9] is supposed to be the self-tuning version (i.e. the version corresponding precisely to the depinning transition) for the models [5, 6]. Therefore one could expect that our modified version of the SOD model, which again includes driving force F, will just reproduce the results in [5, 6]. Indeed, the model also gives the depinned phase with $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$, and the pinned phase with the directed percolation exponents. But the model differs from the original SOD model not only by the inclusion of the driving force F; the important generalization of the SOD growth rule is that the movement of a group of cells of arbitrary size is allowed, if this movement is the most profitable one energetically (3). (With the restriction allowing one to move only a single cell our model is still exactly equivalent to the SOD.) As a consequence, the picture of the phase transition appears to be very different from that of the models in [5, 6]. The average size of groups of moving cells diverges when $F \to F_0$, in drastic contrast with the original SOD model. At the same time the exponents at the transition point appeared to be those of the Parisi models. So we conclude that the collective effects are responsible for the change of the directed percolation exponents [5–9] at the depinning transition to that of the Parisi [3] and the renormalization group [4] values.

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